Two-Loop Beta Function for Bosonic Sigma Model in Ricci-Flat Target Space

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Abstract

This expository paper calculates the beta function of the Bosonic non-linear sigma model with a Ricci-flat Euclidean target space to two-loop order following [AGFM81]. It was written as the final project for Professor Xi Yin's Physics 253b in Spring 2023.

1 Introduction

This paper is about the beta function of the Bosonic non-linear sigma model with a Ricci-flat Euclidean target space and a flat worldsheet. This model is specified by the partition function

$$Z = \int [D\phi] e^{-\frac{1}{h} I[\phi]}$$

where h is some loop counting parameter and $I[\phi]$ is the action defined by

$$I[\phi] = \frac{1}{2} \int d^2x g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j.$$

In the above expression $\phi : W \to M$ should be thought of as a map from a two dimensional worldsheet W to some target manifold M of arbitrary number of dimensions. μ denotes the indices on the worldsheet while i, j are indices on the target M. Note that on the classical level, $I[\phi]$ is independent of coordinates on both the target space and the worldsheet.

Note that as written, the action $I[\phi]$ cannot be dealt with using perturbative QFT due to the ϕ dependence in the metric g_{ij}. Thus, we almost always will perform taylor expansion

$$g_{ij}(\phi) = g_{ij}(\phi_0) + \partial_k g_{ij}(\phi_0)(\phi^k - \phi_0^k) + ...$$

so that $g_{ij}(\phi_0)$ provides the kinetic term and the derivatives such as $\partial_k g_{ij}(\phi_0)(\phi - \phi_0)^k$ are treated as interaction terms for Feynmann diagram calculations.

This paper will be focused on the beta function β_{ij} for the metric g_{ij} for a Ricci-flat target space. It is easy to show that β_{ij} at 1-loop order is proportional to the Ricci tensor R_{ij}. A natural question to consider is what happens to the beta function when the target is already Ricci-flat (in a Calabi-Yau manifold for instance). In the following pages, we will compute β_{ij} to 2-loop order for a Ricci-flat target and show that

$$\beta_{ij} = \frac{h^2}{32\pi^2} R_{iklm} R_j^{\ klm}.$$

2 Definition of the Beta Function and the 1PI RG Procedure

Renormalization can be done in many ways, so it is worthwhile to briefly describe the RG procedure that we will use to determine the beta function. We first define the 1PI effective action $\Gamma[\varphi]$ by

$$\exp(-\frac{1}{h}\Gamma[\varphi]) = \int [D\phi] \exp(-\frac{1}{h}(I[\varphi + \phi] - \int d^2x \phi J)) \tag{1}$$

where J is chosen so that

$$\langle \phi \rangle_J := \int [D\phi] \exp\left(-\frac{1}{h}(I[\varphi + \phi] - \int d^2x \phi J)\right) \phi = 0.$$

Note that in Equation 1, the φ on the left is an abstract map $\varphi : W \to M$ while the right hand side a specific coordinate is invoked to perform the addition $\varphi + \phi$. This coordinate dependence will be discussed in the next section.

With this definition, $\Gamma[\varphi]$ will take the form

$$\Gamma[\varphi] = \int d^2x \frac{1}{2} g_{ij}^{1PI} \partial_\mu \varphi^i \partial^\mu \varphi^j + ...$$

where g_{ij}^{1PI} is defined to be the coefficient for $\frac{1}{2}\partial_\mu \varphi^i \partial^\mu \varphi^j$ appearing in the effective action Γ.

To regularize the UV divergence, we will use dimensional regularization $D = 2 - \epsilon$. In our convention here, $[h] = -2$, $[\varphi] = -1$ and $[d^D x] = -D$ where [] denotes mass dimension. Thus, $[g_{ij}] = -\epsilon$. Using Feynman diagrams, we can compute $g_{ij}^{1PI}(g)$ as a function of our bare coupling g_{ij}. It is expected that g_{ij}^{1PI} is something physical, so it should be finite even when we take our UV regulator off (ϵ going to zero in dimensional regularization).

In the minimal subtraction scheme, we will express the bare coupling g_{ij} as

$$g_{ij} = \mu^{-\epsilon}\left(g^R_{ij} + \sum_{L=1}^{\infty} h^L \sum_{n=1}^{\infty} \frac{1}{\epsilon^n} K^{L,n}_{ij}\right)$$

in terms of the dimensionless renormalized coupling g^R_{ij} where the $K^{L,n}_{ij}$s known as counterterms are functions of g^R_{ij}. $K^{L,n}_{ij}$ can then be determined once we assume that g^R_{ij} is finite and stipulate that g^{1PI}_{ij} is manifestly finite expressed as a function of g^R_{ij}. The beta function is defined by $\beta_{ij} := \frac{dg^R_{ij}}{d\log\mu}$. It will be a function of g^R_{ij} that one can solve for using the condition that the bare coupling does not change: $\frac{dg_{ij}}{d\log\mu} = 0$.

3 Preparation in Differential Geometry

Since we are dealing with a manifold propagating in a potentially curved target, there are some differential geometry preliminaries necessary for the calculation. First of all, we would like our path integral measure $[D\phi]$ to be coordinate independent and based on something intrinsic to our target manifold. The solution is to utilize the geodesic equation. Let $\xi(x) \in T_{\varphi(x)}M$ be a tangent vector at $\varphi(x) \in M$. Let $\phi(\varphi,\xi)(x) \in M$ be the point reached by the geodesic when $t = 1$ starting at $\varphi(x)$ with the initial tangent vector given by $\xi(x)$. We then use $\xi(x)$ to define the fluctuation around the background field $\varphi(x)$. Thus, the covariant version of the 1PI effective action described by Equation 1 is

$$\exp(-\frac{1}{h}\Gamma[\varphi]) = \int [D\xi]\exp(-\frac{1}{h}(I[\phi(\varphi,\xi)] - \int d^2x\xi J)) \tag{2}$$

where J is chosen that

$$\langle\xi\rangle_J := \int [D\xi]\exp\left(-\frac{1}{h}(I[\phi(\varphi,\xi)] - \int d^2x\xi J)\right)\xi = 0.$$

Note that in Equation 2, $\varphi : W \to M$ on the left hand side is an abstract map while the right hand side depends on a specific basis of the tangent space $T_{\varphi(x)}M$. When we perform a change of coordinate $\phi \mapsto \phi'$, the expression in Equation 1 will gain a nonconstant Jacobian $J(\phi')$ and will no longer look like Equation 1. However, for the expression in Equation 2, when we perform a change of coordinate which induces a change of basis on the tangent space, our Jacobian is just a constant J which will be normalized away. Thus, we will use Equation 2 as our definition for the 1PI effective action.

To obtain an explicit expression for $I[\phi(\varphi,\xi)]$, we have to calculate Taylor expansions for $g_{ij}(\phi(\varphi,\xi))$ and

$\phi(\varphi, \xi)$. To obtain the Taylor expansion of $\phi(\varphi, \xi)$, one utilize the geodesic equation

$$\frac{d^2}{dt^2}\lambda^i + \Gamma^i_{jk}(\lambda)(\frac{d}{dt}\lambda^k)(\frac{d}{dt}\lambda^l) = 0.$$

and the definition that $\phi^i = \lambda^i|_{t=1}$. In particular, we have

$$\phi = \lambda(1) = \lambda(0) + \lambda'(0) + \lambda''(0) + ...$$

where $\lambda'(0) = \xi$ and the higher time derivatives can be expressed in terms of ξ by utilizing the geodesic equation and its time derivatives. The expansion for $g_{ij}(\phi)$ can be obtained by first expanding g_{ij} in coordinate and then plugging in the Taylor expansion for $\phi(\varphi, \xi)$ in terms of ξ. For instance, we have

$$g_{ij}(\phi(\varphi, \xi)) = g_{ij}(\varphi) + \frac{1}{3}R_{iklj}\xi^k\xi^l + ...$$

Since we care about the 2-loop result, we only need to expand $I[\phi(\varphi, \xi)]$ to fourth order in ξ since higher order interaction vertices will imply higher number of loops in our Feynmann diagrams. The result of this Taylor expansion, provided in [AGFM81], is

$$I[\phi(\varphi, \xi)] = I[\varphi] + \sum_{i=1}^{\infty} I_i[\varphi, \xi]$$

where I_i, denoting the $i-$th degree piece of $I[\phi(\varphi, \xi)]$, is given by

$$I_2[\varphi, \xi] = \frac{1}{2}\int d^2x\ g_{ij}\nabla_\mu\xi^i\nabla^\mu\xi^j + R_{ik_1k_2j}\xi^{k_1}\xi^{k_2}\partial_\mu\varphi^i\partial^\mu\varphi^j$$

$$I_3[\varphi, \xi] = \frac{1}{2}\int d^2x\ \frac{1}{3}\nabla_{k1}R_{ik_2k_3j}\xi^{k_1}\xi^{k_2}\xi^{k3}\partial_\mu\varphi^i\partial^\mu\varphi^j + \frac{4}{3}R_{ik_1k_2k_3}\xi^{k_1}\xi^{k_2}\nabla_\mu\xi^{k_3}\partial^\mu\varphi^i$$

$$\begin{aligned} I_4[\varphi, \xi] = \frac{1}{2}\int d^2x\ & \frac{1}{2}\nabla_{k_1}R_{ik_2k_3k_4}\xi^{k_1}\xi^{k_2}\xi^{k_3}\nabla_\mu\xi^{k_4}\partial^\mu\varphi^i + \frac{1}{3}R_{k_1k_2k_3k_4}\xi^{k_2}\xi^{k_3}\nabla_\mu\xi^{k_1}\nabla^\mu\xi^{k_4} \\ & + \frac{1}{12}(\nabla_{k_1}\nabla_{k_2}R_{ik_3k_4j} + 4R^m_{k_1k_2i}R_{mk_3k_4j})\xi^{k_1}\xi^{k_2}\xi^{k_3}\xi^{k_4}\partial_\mu\varphi^i\partial^\mu\varphi^j \end{aligned}$$

In the above expression, we used the abbreviation $\nabla_\mu\xi^i := \partial_\mu\xi^i + \Gamma^i_{jk}\xi^j\partial_\mu\phi^k$. Also, notice that we have ignored I_1 since linear terms do not contribute to the computation of $\Gamma[\varphi]$ due to the choice of J in the definition.

We would like to use the $\partial_\mu \xi^i \partial^\mu \xi^j g_{ij}$ part inside I_2 as the kinetic term in our path integral and use Wick contractions to calculate the contribution of other terms. The obvious problem is that $g_{ij}(\varphi(x))$ is non-constant with respect to the worldsheet coordinate x, so we don't have the usual Gaussian integral. The solution to this is to use Vielbein $e^i_a(\varphi)$, which are objects with two indices so that the identity

$$g_{ij} e^i_a e^j_b = \delta_{ab} =: g_{ab}$$

is satisfied. Let e^a_i be the inverse for e^i_a. We define $\xi^a := e^a_i \xi^i$. Then $\partial_\mu \xi^i \partial^\mu \xi^j g_{ij} = \partial_\mu \xi^a \partial^\mu \xi^b g_{ab}$ and $g_{ab} = \delta_{ab}$ is a constant. Due to this change of variable being linear, the Jacobian to adjust $[D\xi^i]$ to $[D\xi^a]$ is a constant and can be ignored after normalization. Note that the other terms in the expansion can also be expressed in terms of ξ^a. For instance $R_{ik_1k_2j}\xi^{k_1}\xi^{k_2}\partial_\mu\varphi^i\partial^\mu\varphi^j = R_{iabj}\xi^a\xi^b\partial_\mu\varphi^i\partial^\mu\varphi^j$ where $R_{iabj} = R_{iklj}e^k_a e^l_b$. The quadratic part becomes

$$I_2[\varphi, \xi] = \frac{1}{2}\int d^2x \; g_{ab}(\partial_\mu \xi^a + \omega^{ac}_i \partial_\mu \phi^i \xi^c)(\partial^\mu \xi^b + \omega^{bc}_i \partial_\mu \phi^i \xi^c) + R_{iabj}\xi^a\xi^b\partial_\mu\varphi^i\partial^\mu\varphi^j$$

where ω^{ab}_i is called the spin connection. Practically, ω can be concretely thought of as some multi-index object that depends on e^a_i and Γ^i_{jk}. When we calculate Feynman diagrams, we will ignore vertices involving ω since it cannot give a covariant (with respect to target space diffeomorphism) contribution to g^{1PI}_{ij} at the two-loop order, so its effect should cancel out in the end. Inverting the kinetic part $\frac{1}{2h}\int d^2x \; g_{ab}\partial_\mu\xi^a\partial^\mu\xi^b$, we obtain our propagator:

$$\langle \xi^a(x)\xi^b(y)\rangle = \int \frac{d^Dk}{(2\pi)^D} e^{ik(x-y)} \frac{hg^{ab}}{k^2+\mu^2}$$

where μ^2 acts as the IR regulator.

4 Diagrams and Calculation

With everything prepared, we will calculate g^{1PI}_{ij} in terms of the bare metric g_{ij} to order h^2. For Feynman diagrams, each vertex gives a factor of $1/h$ while each propagator gives a factor of h. Therefore, order h^2 is equivalent to considering diagrams with no more than two loops.

The definition of $\Gamma[\varphi]$ is given by Equation 2, so the way to calculate g^{1PI}_{ij} will be to look at the right hand side of Equation 2 and collect terms that are proportional to $\partial_\mu\varphi^i\partial^\mu\varphi^j$. These terms can come from I_2, I_3, or I_4 since higher order vertices in ξ will result in more than 2 loops. If we use dashed lines to represent the background field φ and solid lines for the ξ field, we are looking for diagrams with two dashed lines and up to two solid line loops. After some observation, we see that the possible diagrams are given by the following figure.

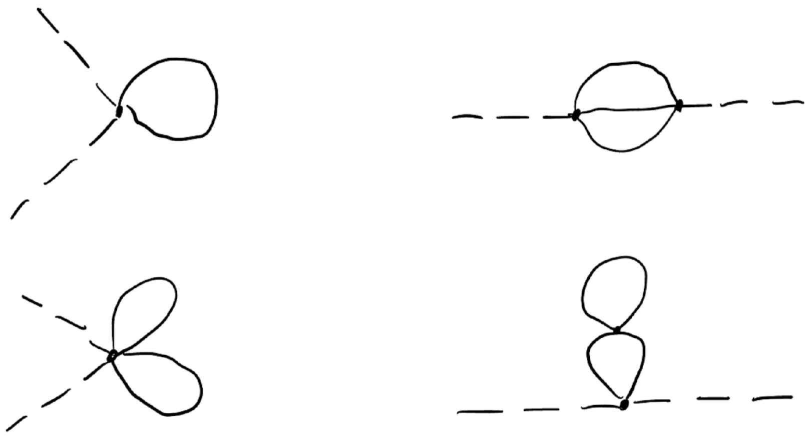

Figure 1: The Feynman Diagrams Contributing to g_{ij}^{1PI}. We will refer to these as the 1-loop diagram, the sunset, the butterfly, and the snowman.

The 1-loop diagram comes from the vertex $R_{iabj}\xi^a\xi^b\partial_\mu\varphi^i\partial^\mu\varphi^j$. Due to the contraction g^{ab} coming from the $\xi^a\xi^b$ propagator, this contribution will be proportional to R_{ij}, which is 0 by our assumption of Ricci-flatness. The snowman diagram is due to the term $\frac{1}{3}R_{k_1k_2k_3k_4}\xi^{k_2}\xi^{k_3}\nabla_\mu\xi^{k_1}\nabla^\mu\xi^{k_4}$ and $R_{iabj}\xi^a\xi^b\partial_\mu\varphi^i\partial^\mu\varphi^j$. Simple observation shows that the only Wick contraction pattern that does not produce the Ricci tensor will still be zero due to integrating an odd function.

Thus, we only need to consider the sunset and the butterfly. Let's consider the sunset diagram first. Expanding the right hand side of Equation 2, we see that the sunset diagrams gives the following contribution to $\Gamma[\varphi]$.

$$-\frac{1}{h}\Gamma[\varphi] \supset \frac{1}{2}(\frac{-4}{3h})^2 \int d^2y_1 d^2y_2 \ \left(R_{iabc}(y_1)\partial^\mu\varphi^i(y_1)\right)\left(R_{i'a'b'c'}(y_2)\partial^\nu\varphi^{i'}(y_2)\right)\langle(\xi^a\xi^b\partial_\mu\xi^c)(y_1)(\xi^{a'}\xi^{b'}\partial_\nu\xi^{c'})(y_2)\rangle$$

We can perform Taylor expansion around y_1 to obtain

$$R_{i'a'b'c'}(y_2)\partial^\nu\varphi^{i'}(y_2) = R_{i'a'b'c'}(y_1)\partial^\nu\varphi^{i'}(y_1) + \partial_\lambda\left(R_{i'a'b'c'}(y_1)\partial^\nu\varphi^{i'}(y_1)\right)(y_2-y_1)^\lambda + ...$$

The derivative terms in this Taylor expansion will not give contribution of the form $\frac{1}{2}\partial_\mu\varphi^i\partial^\mu\varphi^j$ in $\Gamma[\varphi]$ so they are irrelevant for finding the coefficient g_{ij}^{1PI}. Thus, we only consider the constant term in the above

Taylor expansion and arrive at

$$-\frac{1}{h}\Gamma[\varphi] \supset \frac{1}{2}(\frac{-4}{3h})^2 \int d^2y_1 d^2y_2 \ \left(R_{iabc}(y_1)\partial^\mu\varphi^i(y_1)\right)\left(R_{i'a'b'c'}(y_1)\partial^\nu\varphi^{i'}(y_1)\right)\langle(\xi^a\xi^b\partial_\mu\xi^c)(y_1)(\xi^{a'}\xi^{b'}\partial_\nu\xi^{c'})(y_2)\rangle$$
$$\approx \frac{1}{2D}\int d^2y \ hR_{iabc}R_j^{abc}\partial_\mu\varphi^i\partial^\mu\varphi^j\left(\int \frac{d^Dk}{(2\pi)^D}\frac{1}{k^2+\mu^2}\right)^2$$

The $\approx$ sign means having the same singular part (as $\epsilon \to 0$) and since we are calculating the beta function, the singular part is all we care about. To perform the above integral and get that clean expression, we have to consider all six Wick contractions individually and use the following tensor identity:

$$R_{ibac}R_{i'}^{\ bac} = 2R_{iacb}R_{i'}^{\ bca}$$

which can be easily derived using the fact that $R_{i[jkl]} = 0$. Moreover, there are two kinds of momentum integral one needs to perform. The first one is the following.

$$\int \frac{d^Dk_1d^Dk_2}{(2\pi)^{2D}}\frac{k_1\nu k_1\mu}{(k_1^2+\mu^2)(k_2^2+\mu^2)((k_1+k_2)^2+\mu^2)} = \int \frac{d^Dk_1d^Dk_2}{(2\pi)^{2D}}\frac{k_1^2}{(k_1^2+\mu^2)(k_2^2+\mu^2)((k_1+k_2)^2+\mu^2)}\frac{\delta_{\nu\mu}}{D}$$
$$\approx \frac{\delta_{\nu\mu}}{D}\left(\int \frac{d^Dk}{(2\pi)^D}\frac{1}{k^2+\mu^2}\right)^2$$

The first equality above utilizes the worldsheet euclidean symmetry of the expression to rewrite using $\delta_{\nu\mu}$. Recall that $\approx$ sign denotes having the same singular part (as $\epsilon \to 0$). Those two expressions have the same singular part due to the following: the k_2 integral is convergent (since we are in $D = 2-\epsilon$), so after finishing the k_2 integral, the divergence comes from the behavior of $k_1 \to \infty$ and $\frac{k_1^2}{k_1^2+\mu^2} = 1$ in this limit. In practice, this means we can simply camcel out the k_1^2 in the numerator with the $k_1^2+\mu^2$ in the denominator. The second kind of momentum integral we need to perform is

$$\int \frac{d^Dk_1d^Dk_2}{(2\pi)^{2D}}\frac{k_1\nu k_2\mu}{(k_1^2+\mu^2)(k_2^2+\mu^2)((k_1+k_2)^2+\mu^2)} = \int \frac{d^Dk_1d^Dk_2}{(2\pi)^{2D}}\frac{k_1k_2}{(k_1^2+\mu^2)(k_2^2+\mu^2)((k_1+k_2)^2+\mu^2)}\frac{\delta_{\nu\mu}}{D}$$
$$\approx -\frac{\delta_{\nu\mu}}{2D}\left(\int \frac{d^Dk}{(2\pi)^D}\frac{1}{k^2+\mu^2}\right)^2$$

which we obtain by realizing $k_1k_2 = \frac{1}{2}((k_1+k_2)^2 - k_1^2 - k_2^2)$ and canceling out with the denominator. This concludes the calculation techniques needed for the sunset.

The butterfly diagram is much simpler. It is due to the term $4R^m_{k_1k_2i}R_{mk_3k_4j}\xi^{k_1}\xi^{k_2}\xi^{k_3}\xi^{k_4}\partial_\mu\varphi^i\partial^\mu\varphi^j$ in the action. Using the identity $R_{ibac}R_{i'}^{\ bac} = 2R_{iacb}R_{i'}^{\ bca}$ and considering all three contractions give us the result:

$$-\frac{1}{h}\Gamma[\varphi] \supset -\frac{1}{4}\int d^2y R_{iabc}R_j^{\ abc} h\partial_\mu\varphi^i\partial^\mu\varphi^j \left(\int \frac{d^Dk}{(2\pi)^D}\frac{1}{k^2+\mu^2}\right)^2.$$

Putting the results from both diagrams together, we see that

$$g_{ij}^{1PI} = g_{ij} + \mu^{-2\epsilon}h^2 R_{iabc}R_j^{\ abc}(-\frac{1}{32\pi^2\epsilon}) + \text{finite}.$$

In particular, the $\frac{1}{\epsilon^2}$ divergences cancel out between the two diagrams. To make sure g_{ij}^{1PI} is finite, we see that we should substitute in

$$g_{ij} = \mu^{-\epsilon}\left(g_{ij}^R + \frac{h^2}{\epsilon}\frac{1}{32\pi^2}R_{iabc}^R R_j^{R\ abc}\right)$$

where R_{iabc}^R is the Riemann tensor obtained from the renormalized metric g_{ij}^R. We immediately get the result for the beta function shown in the introduction.

References

[AGFM81] Luis Alvarez-Gaume, Daniel Z. Freedman, and Sunil Mukhi. The Background Field Method and the Ultraviolet Structure of the Supersymmetric Nonlinear Sigma Model. *Annals Phys.*, 134:85, 1981.