BGG Correspondence and the Construction of Vector Bundles on Projective Spaces

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December 21, 2023

Abstract

We explore some ways the Bernstein-Gel'fand-Gel'fand (BGG) correspondence can be used to construct vector bundles on projective spaces. To motivate the discussion on vector bundles, we explain the connection between bundles of rank 2 and codimension 2 complete intersections on $\mathbb{P}^n$. Then, we provide an introduction to the BGG correspondence and explain how it leads to the construction of the Tango bundles and the null correlation bundle. The rest of this thesis consists of new results. For $n \geq 3$ and $r \geq n$, we use the correspondence to construct vector bundles of rank r on $\mathbb{P}^n$ with arbitrary homological dimension. Lastly, using results from Popa and Lazarsfeld, for a Kähler threefold with no irregular fibrations, we show that if $\chi(\omega_X) < q(X) - 2$, the cohomology ring $H^*(X, \mathcal{O}_X)$ is generated in degree 0 as a $\bigwedge H^1(X, \mathcal{O}_X)$-module.

1 Introduction

1.1 Vector Bundles on Projective Spaces

The classification of algebraic vector bundles on projective spaces has been an active field. The study of low rank indecomposable bundles has received much attention with interesting open problems such as Hartshorne's conjecture on bundles of rank 2 [16, p.1]. In fact, on $\mathbb{P}^n$, finding any indecomposable bundle of rank less than n is difficult and only a few examples, such as the Tango bundles and the null correlation bundle, are known. A list of all known examples can be found in [17, p.1]. This problem is interesting because there is no good heuristic explanation for why these low rank bundles are rare, yet constructing new examples remains difficult.

When one's goal is classification, it is useful to study objects that cannot be decomposed into smaller subobjects. Thus, special attentions are paid to indecomposable bundles.

Definition 1.1. *A vector bundle $\mathcal{F}$ is indecomposable if there are no non-zero subbundles $\mathcal{F}_1, \mathcal{F}_2$ such that $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$.*

A stronger condition one can impose is simplicity.

Definition 1.2. *A vector bundle $\mathcal{F}$ is simple if $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{F}) = k$ where k is the ground field.*

Note that simplicity implies indecomposability because we can scale the two direct sum components separately if the bundle is decomposable.

As with other areas of mathematics, invariants that partition the objects of study into subgroups are helpful for classification. One invariant for vector bundles is the homological dimension introduced by Bohnhorst and Spindler in [3].

Definition 1.3. *A resolution of a vector bundle $\mathcal{F}$ on projective space is a chain complex of sheaves*

$$\mathcal{C}^\bullet : \quad \cdots \longrightarrow \mathcal{C}^{-1} \longrightarrow \mathcal{C}^0 \longrightarrow 0$$

such that each $\mathcal{C}^i$ is a direct sum of line bundles, $H^i(\mathcal{C}) = 0$ for $i \neq 0$ and $H^0(\mathcal{C}) = \mathcal{F}$. The homological dimension of $\mathcal{F}$, denoted by $hd(\mathcal{F})$ is the shortest length a resolution of $\mathcal{F}$ can have.

Remark 1.4. *We are using cohomological grading for convenience.*

The homological dimension appears in the study of low rank indecomposable bundles due to Corollary 1.7 of [3], which relates the rank $\mathrm{rk}(\mathcal{F})$ with $\mathrm{hd}(\mathcal{F})$ for any indecomposable bundle $\mathcal{F}$ on $\mathbb{P}^n$ through the inequality

$$\mathrm{rk}(\mathcal{F}) \geq n + 1 - \mathrm{hd}(\mathcal{F}).$$

Using Horrock's splitting criterion, one can show that for $\mathcal{F}$ on $\mathbb{P}^n$, $\mathrm{hd}(\mathcal{F}) = 0, 1, ...,$ or $n-1$, where $\mathrm{hd}(\mathcal{F}) = 0$ is equivalent to $\mathcal{F}$ being a direct sum of line bundles [3]. In 2012, Jardim and Prata constructed rank-n simple vector bundles on $\mathbb{P}^n$ of homological dimensions $1, 2, ...,$ and $n-1$, proving that for rank-n bundles, all homological dimensions are possible [17, Theorem 1.3]. They proved this result by performing induction on the homological dimension using Theorem 4.3 in Brambilla's [4]. However, it remained unclear whether the same was true for ranks greater than n and whether having high ranks forbids certain homological dimensions from happening.

The study of vector bundles on projective space is a vast subject and one can ask many interesting questions beyond the rank, indecomposability, simplicity, and homological dimension of bundles. However, these are the main concepts that this thesis is concerned with. To get an overview about other topics such as the stability and moduli of vector bundles on projective space, the reader can start with [20].

1.2 Bernstein-Gel'fand-Gel'fand Correspondence

One useful technique for studying sheaves on projective spaces is the Bernstein-Gel'fand-Gel'fand (BGG) correspondence introduced in [2], which given a $\bigwedge V-$module P where V is a finite dimensional vector space, constructs a chain complex $L(P)$ of free $\mathrm{Sym}V^*$-modules. Sheafification produces a complex $\tilde{L}(P)$ of vector bundles on $\mathbb{P}^n = \mathrm{Proj}(\mathrm{Sym}V^*)$. Therefore, complexes of sheaves become related to modules over $\bigwedge V$. For instance, Eisenbud, Fløystad, and Schreyer constructed the Beilinson monad using the correspondence [10]. Moreover, by restricting to certain kinds of $\bigwedge V$-modules, one can ensure that $\tilde{L}(P)$ has cohomologies that are vector bundles. Coandă and Trautmann considered complexes of $\bigwedge V$-modules that, through the BGG correspondence, gave rise to stable vector bundles [7]. Alternatively, results on vector bundles can produce insights on modules over $\bigwedge V$: Popa and Lazarsfeld discovered Hodge number inequalities by applying the BGG correspondence to cohomology rings [19]. This thesis will explore some ways the BGG correspondence can give rise to interesting vector bundles.

1.3 Thesis Content

Besides the expository sections, the thesis contains some original research. After proving Lemma 3.10 about the exactness of linear free complexes, we are able to generalize the result of [17] to arbitrary ranks larger than or equal to n using the BGG correspondence, showing that all homological dimensions are possible.

Theorem 1.5. *Let $n \geq 3$ and let k be an algebraically closed field. For $l = 1, 2, ..., n-1$ and any $r \geq n$, there exists a simple vector bundle of rank r and homological dimension l on $\mathbb{P}^n_k$, the n-dimensional projective space over k.*

As for low rank bundles, we use the BGG correspondence to construct the Tango bundles and the null correlation bundle, which are among the very few known types of the indecomposable bundles rank less than n on $\mathbb{P}^n$. We prove these bundles are simple using only BGG, without appealing to the usual arguement in literature involving Chern classes and short exact sequences. This argument based on BGG alone has not appeared in any literature that the author knows of.

In a different direction, using similar techniques, we obtain a preliminary result on the numerical invariants of compact Kähler manifolds related to Conjecture 3.9 of [19].

Proposition 1.6. *Let X be an irregular Kähler threefold such that there is no map $f : X \to Y$ with positive dimensional fibers onto a normal analytic variety Y with the property that (any smooth model of) Y has maximal Albanese dimension. Suppose that $q(X) > 4$ and $\chi(\omega_X) < q(X) - 2$. Then, the cohomology ring $H^*(X, \mathcal{O}_X)$ is generated in degree 0 as a $\bigwedge H^1(X, \mathcal{O}_X)$-module.*

The thesis is organized as follows. Section 2 motivates the study of vector bundles on projective spaces by explaining how Hartshorne's conjecture on indecomposable bundles of rank 2 is related to complete intersections of codimension 2 on projective spaces. Section 3 offers a self-contained introduction to the BGG correspondence, how it relates to vector bundles, and the proof of Lemma 3.10 on linear resolutions. Section 4 explains how the famous Tango bundles and null correlation bundle can be constructed using the BGG correspondence. Section 5 presents the proof of Theorem 1.5. Lastly, Section 6 proves Proposition 1.6.

Acknowledgements

First and foremost, I have to thank Professor Mihnea Popa for supervising my research. This project would not have been possible without him. I must thank Wanchun Shen and Duc Vo. They helped me understand the key literature in the area by guiding me in the Directed Reading Program and provided valuable insights when I discussed ideas with them. I thank Professor Marcos Jardim for answering my questions about his paper, which directly inspired my research [17]. I am grateful to Sung Gi Park, who helped me understand some of Hartshorne's arguements. Lastly, I have to thank Professor Noam Elkies for providing a counterexample that saved me from a wrong research approach.

2 Hartshorne's Conjecture on Rank 2 Bundles

Part of the motivation for studying vector bundles on projective space is due to Hartshorne's conjecture on complete intersections [14, p.1017]:

Conjecture 2.1. *If Y is a nonsingular subvariety of dimension r in $\mathbb{P}^n$ and $r > \frac{3}{2}n$, then Y is a complete intersection.*

If we restrict to the case when $r = n-2$ and work over $\mathbb{C}$, Conjecture 2.1 becomes related to rank 2 vector bundles:

Theorem 2.2. *Let $n \geq 7$. Then there exists a codimension 2 smooth $Y \subset \mathbb{P}^n$ that is not a complete intersection if and only if there exists a indecomposable vector bundle of rank 2.*

Intuitively, given a general global section s on some vector bundle of rank 2, we can construct $V(s) \subset \mathbb{P}^n$ where s vanishes. $V(s)$ will have codimension 2 because locally the vanishing of s gives two constraints. The difficulty mainly lies in constructing a vector bundle from a codimension 2 subvariety. We follow the proof appearing in [14], adding in details that Hartshorne felt too trivial to include. We will need two lemmas in the proof.

Lemma 2.3. *Let f_1, f_2 form a regular sequence in an integral local ring A. Let*

$$0 \longrightarrow A \longrightarrow M \xrightarrow{\phi} (f_1, f_2) \longrightarrow 0$$

be a non split exact sequence of A modules. Then $M \cong A^2$

Proof. Let $g_1, g_2 \in M$ be any preimage of f_1, f_2 respectively and $(g_1, g_2) \subset M$ be the submodule generated by g_1, g_2. We have a quotient map $q : A^2 \to (g_1, g_2)$ where the two basis elements e_1, e_2 are mapped to g_1, g_2 respectively. Because f_1, f_2 form a regular sequence, $\ker \phi \circ q = A \cdot (f_2 e_1 - f_1 e_2)$. Then

$$\ker \phi \cap (g_1, g_2) \cong (A \cdot (f_2 e_1 - f_1 e_2)) / \ker q.$$

Because A is torsion free, $\ker \phi \cap (g_1, g_2)$ must be too, so $\ker q = 0$ or $\ker q = A \cdot (f_2 e_1 - f_1 e_2)$. In the first case, we see that $M = (g_1, g_2) = A^2$. The second case is impossible since it implies the exact sequence splits. $\square$

Lemma 2.4. *Let E be a globally generated vector bundle on $\mathbb{P}^n$ and $s \in H^0(\mathbb{P}^n, E)$ be a general section, then $V(s)$ is smooth.*

Proof. Let r be the rank of E and $m = h^0(\mathbb{P}^n, E)$. There exists a map $f : \mathbb{P}^n \to \mathrm{Gr} = \mathrm{Gr}(m-r, m)$ defined by $p \mapsto \ker(H^0(\mathbb{P}^n, E) \to E|_p)$. Let $X_s \subset \mathrm{Gr}$ be the collection of subspaces that contains $s \in H^0(\mathbb{P}^n, E)$. It is clear that $X_s \cong \mathrm{Gr}(m-r, m-1)$ so it is smooth. Moreover,

$$V(s) = X_s \times_{\mathrm{Gr}} \mathbb{P}^n.$$

Kleiman-Bertini theorem then gives us the result [11, B.9.2]. $\square$

Proof of Theorem 2.2. " $\implies$ ": We will first construct a indecomposable vector bundle assuming such variety Y exists. Let $Y \subset \mathbb{P}^n$ be a smooth codimension 2 subvariety that is not a complete intersection. By Theorem 2.2d of [14], there exists integer k such that $\omega_Y = \mathcal{O}_Y(k)$. Let $L = \mathcal{O}_{\mathbb{P}^n}(-k-n-1)$, I_Y be the ideal sheaf for Y, and $j : Y \to \mathbb{P}^n$ be the inclusion. We then have the following equalities in the derived category of sheaves on $\mathbb{P}^n$

$$\begin{aligned} R\mathcal{H}om_{\mathbb{P}^n}(j_*\mathcal{O}_Y, L) =& R\mathcal{H}om_{\mathbb{P}^n}(j_*\mathcal{O}_Y, \omega_{\mathbb{P}^n}(-k)) && (1)\\ =& R\mathcal{H}om_{\mathbb{P}^n}(j_*\mathcal{O}_Y(k), \omega_{\mathbb{P}^n}) && (2)\\ =& R\mathcal{H}om_{\mathbb{P}^n}(j_*\mathcal{O}_Y(k), \omega^\bullet_{\mathbb{P}^n}[-n]) && (3)\\ =& j_*R\mathcal{H}om_Y(\mathcal{O}_Y(k), \omega^\bullet_Y[-n]) && (4)\\ =& j_*R\mathcal{H}om_Y(\mathcal{O}_Y(k), \omega_Y[-2]) && (5)\\ =& j_*R\mathcal{H}om_Y(\mathcal{O}_Y, \mathcal{O}_Y)[-2] && (6)\\ =& j_*\mathcal{O}_Y[-2] && (7) \end{aligned}$$

where $\omega^{n\bullet}_{\mathbb{P}}$ denotes the dualizing complex for $\mathbb{P}^n$. The above equalities are due to the following facts:

1. $R\mathcal{H}om(\mathcal{F}, \mathcal{G}(-d)) = R\mathcal{H}om(\mathcal{F}(d), \mathcal{G})$. This is directly from the definition of derived functors. This explains the second equality.

2. The dualizing complex $\omega^\bullet_X$ on a smooth variety X of dimension n is $\omega_X[n]$ where ω_X is the canonical sheaf. This explains the third and the fifth equality.

3. Coherent duality states that there exists a functor $j^! : D^+_{QCoh}(\mathbb{P}^n) \to D_{QCoh}(Y)$ such that

$$Rj_*R\mathcal{H}om_{\mathcal{O}_Y}(K, j^!M) \cong R\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(Rj_*K, M)$$

   for any $K \in D^-_{Coh}(Y), M \in D^+_{QCoh}(\mathbb{P}^n)$. Moreover, $j^!\omega^\bullet_{\mathbb{P}^n} = \omega^\bullet_Y$ [21, Tag 0AU3] . The fourth equality follows from plugging in $K = \mathcal{O}_Y$ and $M = \omega^\bullet_{\mathbb{P}^n}$ and the fact that $j_* = Rj_*$ because j is affine.

From our calculation of $R\mathcal{H}om$, we obtain immediately that

$$\mathcal{E}xt^0(j_*\mathcal{O}_Y, L) = \mathcal{E}xt^1(j_*\mathcal{O}_Y, L) = 0, \qquad \mathcal{E}xt^2(j_*\mathcal{O}_Y, L) = j_*\mathcal{O}_Y.$$

Now, considering the local-to-global Ext spectral sequence, we obtain the following terms on the E_2 page

$$\begin{aligned} E_2^{1,1} &= H^1(\mathbb{P}^n, \mathcal{E}xt^1(j_*\mathcal{O}_Y, L)) = 0 \\ E_2^{2,0} &= H^2(\mathbb{P}^n, \mathcal{E}xt^0(j_*\mathcal{O}_Y, L)) = 0 \\ E_2^{0,2} &= H^0(\mathbb{P}^n, j_*\mathcal{O}_Y) = \mathbb{C}. \end{aligned}$$

The E_0 page of the spectral sequence has vertical differential maps, so the differential maps on the E_2 page goes from $E_2^{p,q}$ to $E_2^{p+2,q-1}$ (this often omitted fact is true for Grothendieck spectral sequences in general). Therefore $E_\infty^{0,2} = E_2^{0,2}$. $\mathrm{Ext}^2(j_*\mathcal{O}_Y, L)$ is supposed to be filtered by $E_\infty^{0,2}, E_\infty^{1,2}, E_\infty^{2,0}$, so we obtain finally

$$\mathrm{Ext}^2(j_*\mathcal{O}_Y, L) \cong H^0(\mathbb{P}^n, j_*\mathcal{O}_Y) = \mathbb{C}.$$

By considering the long exact sequence obtained from the short exact sequence

$$0 \longrightarrow I_Y \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_Y \longrightarrow 0,$$

we see that

$$\mathrm{Ext}^2(j_*\mathcal{O}_Y, L) \cong \mathrm{Ext}^1(I_Y, L) \qquad \mathcal{E}xt^2(j_*\mathcal{O}_Y, L) \cong \mathcal{E}xt^1(I_Y, L)$$

Under the above identifications, $1 \in H^0(\mathbb{P}^n, j_*\mathcal{O}_Y)$ can be identified with an extension class $f \in \mathrm{Ext}^1(I_Y, L)$ and a global section $s \in H^0(\mathbb{P}^n, \mathcal{E}xt^1(I_Y, L))$. Let

$$0 \longrightarrow L \longrightarrow F \longrightarrow I_Y \longrightarrow 0$$

be the extension defined by f. We will next show that F is a indecomposable rank 2 vector bundle.

First, we show that F is locally free of rank 2. The extension at each stalk

$$0 \longrightarrow L_p \longrightarrow F_p \longrightarrow I_{Y,p} \longrightarrow 0$$

is given by

$$s_p \in \mathcal{E}xt^1(I_Y, L)_p = \mathrm{Ext}_{\mathcal{O}_{\mathbb{P}^n,p}}(I_{Y,p}, L_p).$$

Because s corresponds to $1 \in H^0(\mathbb{P}^n, j_*\mathcal{O}_Y)$, $s_p \neq 0$ if and only if $p \in Y$. Thus, $F_p = L_p \oplus I_{Y,p} = \mathcal{O}_{\mathbb{P}^n,p}^{\oplus 2}$ for $p \notin Y$. Next, we turn to the case when $p \in Y$. The stalk exact sequence no longer splits. Also, Y being smooth implies $\mathcal{O}_{Y,p}$ is a local complete intersection ring, so there exists regular sequence $f_1, f_2 \subset \mathcal{O}_{\mathbb{P}^n,p}$ such that $I_{Y,p} = (f_1, f_2)$. Lemma 2.3 says that $F_p = \mathcal{O}_{\mathbb{P}^n,p}^{\oplus 2}$ once again. Thus, F is locally free of rank 2.

Lastly, we show that F is indecomposable. The composition $F \to I_Y \to \mathcal{O}_{\mathbb{P}^n}$ has a transpose $t : \mathcal{O}_{\mathbb{P}^n} \to F^\vee$. Then, it is clear that $Y = V(t)$. Thus, F being the direct sum of two line bundles implies that $Y = V(t)$ is a complete intersection, violating the original assumption.

" $\Longleftarrow$ ": Let E be a rank 2 indecomposable vector bundle. By twisting, we can assume that E is globally generated. Lemma 2.4 then provides us with a section s and a smooth $Y := V(s)$. It remains to show the Y is not a complete intersection. Taking the transpose of $s : \mathcal{O}_{\mathbb{P}^n} \to E$ gives us an exact sequence

$$0 \longrightarrow L \xrightarrow{c} E^\vee \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_Y \to 0.$$

where $L := \wedge^2 E^\vee$ and c is contraction with s.

Because $j^! = R\mathcal{H}om(j_*\mathcal{O}_Y, \bullet)$ in the case of closed immersion and $j^!\omega_{\mathbb{P}^n}^\bullet = \omega_Y^\bullet$ [21, Tag 0AU3], we have

$$\omega_Y = \mathcal{E}xt^2(j_*\mathcal{O}_Y, \omega_{\mathbb{P}^n}) = L^\vee \otimes \mathcal{O}_Y \otimes \omega_{\mathbb{P}^n}$$

where the second equality is obtained by calculation $\mathcal{E}xt$ with the resolution of $\mathcal{O}_Y$ shown above. By Theorem 2.2d of [14], we can find k such that $\omega_Y = \mathcal{O}_Y(k)$. Then, $L = \mathcal{O}_{\mathbb{P}^n}(-k-n-1)$.

Suppose for contradiction that Y is a complete intersection. Then we can find a decomposable rank 2 bundle E' and a global section s' such that $Y = V(s')$. Repeating the construction above gives us an extension

$$0 \longrightarrow L \longrightarrow E'^\vee \longrightarrow I_Y \longrightarrow 0$$

Thus, $E'^\vee$ and $E^\vee$ are two distinct extension classes in $\mathrm{Ext}^1(I_Y, L)$. However, we showed via a spectral sequence earlier that $\mathrm{Ext}^1(I_Y, L)$ is one dimensional, so there is a contradiction.

$\square$

3 BGG Correspondence

This section provides an introduction to the main tool of this thesis, the BGG correspondence. We also explain how vector bundles can arise as BGG-sheaves and prove Lemma 3.10, which would be crucial to proving the simplicity of vector bundles in later sections.

3.1 BGG Complex

Let k be an algebraically closed field. Let V be an $n+1$-dimensional k-vector space. Let $E = \bigwedge V$ be its exterior algebra. In this paper, for convenience, we assume E is graded positively, i.e., the degree of any $v \in V$ is 1. Let P be a graded left E-module. Let $\{e_0, ..., e_n\} \subset V$ be a basis and $\{x_0, ..., x_n\} \subset V^*$ be the corresponding dual basis. Let $S = \mathrm{Sym} V^* = k[x_0, ..., x_n]$. The following definition of the BGG complex is from [9].

Definition 3.1. *Let $P = \oplus_{i \in \mathbb{Z}} P_i$ be a graded E-module. Then, the BGG complex $L(P)$ is given by*

$$\cdots \longrightarrow S[i] \otimes_k P_i \longrightarrow S[i+1] \otimes_k P_{i+1} \longrightarrow \cdots$$

where the differential δ is defined by

$$\delta : \ 1 \otimes p \mapsto \sum_{i=0}^{n} x_i \otimes e_i p.$$

Using $v \wedge u = -u \wedge v$ and $v \wedge v = 0$ for all $u, v \in V$, one can explicitly check that the differentials compose to zero, so $L(P)$ is a chain complex. We call a chain complex of

$$\cdots \longrightarrow M^i \longrightarrow M^{i+1} \longrightarrow \cdots$$

of graded S-modules a linear free complex if each M^i appearing in it is free and the generators of M^i have degree $-i$. Note that the cohomological grading of the chain complex and the positive grading for $v \in V$ we are using here are different from the convention of [9]. The BGG correspondence then says the following.

Proposition 3.2. *L is an equivalence of categories from graded E-modules to linear free complexes over S.*

Proof. We can construct an explicit inverse. Given a linear free complex $C^\bullet$, then $C^i = S[i] \otimes P_i$ for some $k-$vector space P_i. Because the differentials $\delta : S[i] \otimes P_i \to S[i+1] \otimes P_i$ preserves grading,

$$\delta|_{1 \otimes P_i} : 1 \otimes P_i \to V^* \otimes P_{i+1}.$$

Given a $v \in V$, it defines a contraction $c_v : V^* \otimes P_{i+1} \to P_{i+1}$. Let $P = \bigoplus_i P_i$. We define a V action on P

by

$$vp := c_v(\delta(1 \otimes p))$$

where $p \in P_i$ and $vp \in P_{i+1}$ It is straightforward to check that associativity holds, so P is a $\bigotimes V$-module. Using the fact that $\delta \circ \delta = 0$ and expanding in basis, one sees that $v(v(p)) = 0$ for all $v \in V$. Thus, P is a well-defined $E-$module. We then define $L^{-1}(C^\bullet) := P$. Checking that L^{-1} is the inverse and its functoriality is straightfoward from its definition.

□

This version of the correspondence is presented in [9]. It is worth noting that the BGG correspondence can also be stated as an equivalence of categories between the derived category of S-modules and the derived category of E-modules as shown in Corollary 2.7 of [10]. However, we will not be needing this derived equivalence here.

Example 3.3. *The complex $L(E)$ is the dual of the Koszul complex for the sequence $(x_0, x_1, ..., x_n)$. Given the graded S-module $S[i] \otimes P_i$, its dual*

$$\mathrm{Hom}_{SMod}(S[i] \otimes P_i, S) \cong S[-i] \otimes P_i^*$$

is also a graded S-module where the grading is defined by the shift in degree of the module homomorphism. Therefore, using $(\bigwedge^l V)^* = \bigwedge^l (V^*)$, we have

$$\mathrm{Hom}(L(E), S) : \quad 0 \to S[-n-1] \otimes \bigwedge^{n+1} V^* \to S[-n] \otimes \bigwedge^{n} V^* \cdots \to S[0] \otimes k \to 0.$$

Here every differential map

$$\mathrm{Hom}(\delta, S) : S[-l-1] \otimes \bigwedge^{l+1} V^* \to S[-l] \otimes \bigwedge^{l} V^*$$

is defined by pullback. Calculating the pullback using our explicit basis of $\{x_0, ..., x_n\}$ arrives at the expression:

$$1 \otimes (x_{i_0} \wedge x_{i_1} \wedge \cdots \wedge x_{i_l}) \mapsto \sum_{j=0}^{l} (-1)^j x_{i_j} \otimes (x_{i_0} \wedge \cdots \wedge \hat{x}_{i_j} \wedge \cdots \wedge x_{i_l})$$

where $\hat{x}_{i_j}$ denotes omission. This is the Koszul complex and it is only nonexact at the term $S[0] \otimes k$. Using the fact that the Koszul complex is isomorphic to its own dual, we see that $L(E)$ is exact except at $S[n+1] \otimes \bigwedge^{n+1} V$ [9, p. 126].

3.2 BGG-Sheaves and Faithful Modules

In the rest of the paper, all the E-modules we consider will be such that $c = \max_i(P_i \neq 0)$ is well defined. We can sheafify $L(P)$ to produce the complex $\tilde{L}(P)$ given by

$$\cdots \longrightarrow P_i \otimes_k \mathcal{O}_{\mathbb{P}^n}(i) \xrightarrow{d_i} P_{i+1} \otimes_k \mathcal{O}_{\mathbb{P}^n}(i+1) \longrightarrow \cdots$$

To go back to modules, we will make use of the functor Γ_* that turns sheaves into modules.

Definition 3.4. *The functor $\Gamma_* : QC(\mathbb{P}^n) \to SMod^{gr}$ from quasi-coherent sheaves to graded S-modules is defined by $\Gamma_*(\mathcal{F}) = \bigoplus_{i\in\mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{F}(i))$. The ith graded piece of $\Gamma_*(\mathcal{F})$ is defined to be $H^0(\mathbb{P}^n, \mathcal{F}(i))$.*

It follows immediately that $L(P) = \Gamma_*(\tilde{L}(P))$. Following [19], we call the cohomology at the last nonzero term the BGG-sheaf.

Definition 3.5. *The BGG-sheaf refers to $H^c(\tilde{L}(P))$ where $c = max_i(P_i \neq 0)$.*

If one is concerned with vector bundles, one might consider two questions about this construction:

1. When is a vector bundle on $\mathbb{P}^n$ the BGG-sheaf of some module P?
2. How can we characterize modules whose BGG complexes provide resolutions for vector bundles?

These questions have simple answers which this subsection presents.

Proposition 3.6. *Let $\mathcal{F}$ be any vector bundle on $\mathbb{P}^n$, then there is a module P whose BGG-sheaf is $\mathcal{F}$ such that $\tilde{L}(P)$ is a resolution for $\mathcal{F}$.*

Proof. Applying Serre vanishing, we can assume that $\mathcal{F}(n)$ is 0-regular in the sense of Castelnuovo-Mumford. Thus, $\mathcal{F}$ has a linear resolution

$$\mathcal{C}^\bullet : \ \cdots \to P_{-2} \otimes \mathcal{O}_{\mathbb{P}^n}(-2) \to P_{-1} \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \to P_0 \otimes \mathcal{O}_{\mathbb{P}^n} \to 0$$

where P_i are finite dimensional k-vector spaces [18, Prop 1.8.8]. Then, $\Gamma_*(\mathcal{C}^\bullet)$ is a linear free complex, and the module P we are looking for is simply

$$P = \bigoplus P_i = L^{-1}(\Gamma_*(\mathcal{C}^\bullet))$$

where L^{-1} is the inverse functor based on the equivalence of categories shown in Proposition 3.2.

□

To articulate when the complex $\tilde{L}(P)$ provides a resolution, Bernstein, I. Gel'fand, and S. Gel'fand introduced the notion of faithful modules.

Definition 3.7. *P is faithful if for all $v \in V$, the following sequence of k-vector spaces*

$$\cdots \longrightarrow P_{i-1} \xrightarrow{\cdot v} P_i \xrightarrow{\cdot v} P_{i+1} \longrightarrow \cdots$$

is exact at all $i < c$ where P_i is the ith graded piece of P, $c = \max_i(P_i \neq 0)$, and $\cdot v$ denotes multiplication by v due to the E-module structure.

Proposition 3.8. *P is faithful if and only if the BGG sheaf is a vector bundle and $\tilde{L}(P)$ gives a resolution for it.*

Proof. We will need a basis-free description of the differential maps d_is appearing in $\tilde{L}(P)$. Let

$$d_i(-i-1) : P_i \otimes_k \mathcal{O}_{\mathbb{P}^n}(-1) \to P_{i+1} \otimes_k \mathcal{O}_{\mathbb{P}^n}$$

be the map obtained by twisting d_i. Over $[v] \in \mathbb{P}^n$, the fiber has the form

$$(P_i \otimes_k \mathcal{O}_{\mathbb{P}^n}(-1))\,|_{[v]} = P_i \otimes kv$$

because $\mathcal{O}_{\mathbb{P}^n}(-1)$ is the tautological bundle. In this description, one can easily check by restricting to affine charts that the map

$$d_i(-i-1)|_{[v]} : P_i \otimes kv \to P_{i+1}$$

is given by

$$p \otimes v \mapsto vp$$

where we are invoking the E-module action. In fact, this is the definition given in the original BGG paper [2].

"$\Longrightarrow$": Let P be a faithful module. First, we observe that $\operatorname{rank}(\cdot v : P_i \to P_{i+1})$ is lower semicontinuous as a function of v for all i using the vanishing off determinants. Suppose for contradiction that the rank is not constant across different vs for some i, then $\dim\ker(\cdot v : P_i \to P_{i+1})$ is a non-constant upper semicontinuous function. Thus, $\operatorname{rank}(\cdot v : P_{i-1} \to P_i)$ is a non-consant upper semicontinuous function, contradicting that it is lower semicontinous.

Thus, each d_i is a constant rank map of vector bundles, so the BGG-sheaf $\mathcal{F}$ must also be a vector bundle.

Moreover, based on the basis-free description of d_i shown above, it is clear that $\tilde{L}(P)$ is a resolution for $\mathcal{F}$ in the category of vector bundles, so it is also a resolution in the category of quasicoherent sheaves.

" $\Longleftarrow$ ": Immediately follows from the previous basis-free description of the differential.

$\square$

3.3 Simple Module Implies Simple Bundle

In this subsection, we prove a lemma that allows us to translate the simplicity of many E-modules to the simplicity of their BGG-sheaves. It will be the main tool for us to prove the simplicity of vector bundles in later sections. We will use $\tilde{M}$ or $M^{\sim}$ to denote the sheaf on $\mathbb{P}^n$ obtained by sheafifying a graded S-module M. One property about Castelnuovo-Mumford regularity that will be crucial to us is the following [9, Proposition 4.16].

Proposition 3.9. *Let M be a finitely generated graded S-module that is d-regular. Then the canonical map*

$$M_d \to \Gamma_*(\tilde{M})_d$$

is surjective.

In general, an exact sequence of sheaves does not produce an exact sequence of global sections. However, knowing the exactness of $\tilde{L}(P)$ says a lot about the exactness of $L(P)$:

Lemma 3.10. *Let $P = \bigoplus_{i=0}^{c} P_i$ be an E-module such that $c := \max_i(P_i \neq 0) \in \{0, 1, ..., n\}$ and $H^i(\tilde{L}(P)) = 0$ for all $i \neq c$. Then, $L(P)$ is the minimal free resolution for $\Gamma_*(\mathcal{F})_{\geq -c}$, where $\mathcal{F}$ is the BGG-sheaf of P.*

Proof. Note that once we prove that $L(P)$ is a resolution, the fact that it is minimal is automatic since $L(P)$ is a linear free complex. We will prove the lemma by performing induction on c. The case for $c = 0$ is clear. Now, let $c = 1, ..., n$ and assume the lemma is already true for $c - 1$. In particular, the lemma applies for $P_{\leq c-1}$. To obtain the lemma for c, we will need to prove:

- $H^{c-1}(L(P)) = 0$, i.e. we have exact S-module maps at

$$S[c-2] \otimes P_{c-2} \xrightarrow{\Gamma_* d_{c-2}} S[c-1] \otimes P_{c-1} \xrightarrow{\Gamma_* d_{c-1}} S[c] \otimes P_c \tag{8}$$

  since the exactness for $0, 1, ..., c-2$ is gauranteed by the induction hypothesis;

- the exactness of

$$S[c-1] \otimes P_{c-1} \xrightarrow{\Gamma_* d_{c-1}} S[c] \otimes P_c \to \Gamma_*(\mathcal{F})_{\geq -c} \to 0. \tag{9}$$

We will first show the exactness of (8). We have the following commutative diagram

$$\begin{tikzcd} P_{c-1} \otimes \mathcal{O}(c-1) \arrow[rr, "d_{c-1}"] \arrow[dr, "f"'] & & P_c \otimes \mathcal{O}(c) \\ & \operatorname{coker} d_{c-2} \arrow[ur, "\iota"'] & \end{tikzcd}$$

where f is the natural cokernal map and ι is an inclusion of sheaves due to the exactness of $\tilde{L}(P)$ at $c-1$. Note that we are using $\mathcal{O}(c)$ to denote $\mathcal{O}_{\mathbb{P}^n}(c)$ and the tensor product is over k. We then have

$$\ker \Gamma_* d_{c-1} = \ker \Gamma_* \iota \circ \Gamma_* f = \ker \Gamma_* f = \operatorname{im} \Gamma_* d_{c-2}$$

where the first equality is due to the functoriality of Γ_*, the second equality is because $\Gamma_* \iota$ is an injection due to the left exactness of Γ_*, and the third equality is by applying the induction hypothesis on $P_{\leq c-1}$ because $\operatorname{coker} d_{c-2}$ is the BGG-sheaf of $P_{\leq c-1}$. This shows the exactness of (8).

Next, we show the exactness of (9). Let $M := H^c(L(P))$. Because the sheafification functor $(-)^\sim : M \mapsto \tilde{M}$ is exact, we know that $\tilde{M} \cong \mathcal{F}$. Therefore, we have a natural map of graded S-modules

$$\phi : M \to \Gamma_* \mathcal{F}.$$

It is obvious that $M = M_{\geq -c}$ because it is a quotient of $P_c \otimes S[c]$ so all the generators are of degree $-c$. The exactness of (9) is then equivalent to ϕ being an isomorphism in degrees $\geq -c$, with the exactness at $S[c] \otimes P_c$ being equivalent to the injectivity of ϕ and the exactness at $(\Gamma_* \mathcal{F})_{\geq -c}$ equivalent to the surjectivity of ϕ.

Let's show surjection of $\phi : M \to \Gamma_* \mathcal{F}_{\geq -c}$ first. M has a linear resolution given by $L(P)$ and is generated by degree $-c$ elements, so it is $-c$-regular. Therefore, it is d-regular for any $d \geq -c$. Applying Proposition 3.9, we immediately obtain the surjection.

Next, we turn to showing injection for $\phi : M_{\geq -c+1} \to (\Gamma_* \mathcal{F})_{\geq -c+1}$ before proving injection in degree $-c$ as well. We have the following exact sequence:

$$0 \to \operatorname{coker} d_{c-2} \to P_c \otimes \mathcal{O}(c) \to \mathcal{F} \to 0.$$

By the left exactness of Γ_*, we have injection $\iota : \Gamma_*(P_c \otimes \mathcal{O}(c))/(\Gamma_* \operatorname{coker} d_{c-2}) \to \Gamma_* \mathcal{F}$. The following diagram clarifies the algebra happening.

$$\begin{tikzcd}
 & & & & \Gamma_*\mathcal{F} \\
\Gamma_*(P_{c-2}\otimes\mathcal{O}(c-2)) \arrow[r, "\Gamma_* d_{c-2}"] & \Gamma_*(P_{c-1}\otimes\mathcal{O}(c-1)) \arrow[rr, "\Gamma_* d_{c-1}"'] \arrow[dr] & & \Gamma_*(P_c\otimes\mathcal{O}(c)) \arrow[r] \arrow[ur] & M \arrow[u, "\phi"'] \\
 & & \Gamma_*\text{coker } d_{c-2} \arrow[ur, "\subset"] & &
\end{tikzcd}$$

In the diagram, both the horizontal maps and the diagonal maps are exact at $\Gamma_*(P_c \otimes \mathcal{O}(c))$. From the diagram, we see that $\operatorname{im}\Gamma_* d_{c-1} \subset \Gamma_* \operatorname{coker} d_{c-2}$. Because $\operatorname{coker} d_{c-2}$ is the BGG-sheaf of $P_{\leq c-1}$, the induction hypothesis tells us that

$$(\Gamma_* \operatorname{coker} d_{c-2})_{\geq -c+1} = \operatorname{coker}\Gamma_* d_{c-2} = \operatorname{im}\Gamma_* d_{c-1}.$$

The original map ϕ can be factored using $\phi = \iota \circ q$ where q is the natural quotient as shown in the diagram below.

$$\begin{tikzcd}
M \arrow[r, "\cong"] \arrow[d, "\phi"'] & \Gamma_*(P_c\otimes\mathcal{O}(c))/\text{im } \Gamma_* d_{c-1} \arrow[d, "q"] \\
\Gamma_*\mathcal{F} & \Gamma_*(P_c\otimes\mathcal{O}(c))/\Gamma_*\text{coker } d_{c-2} \arrow[l, "\iota"', hook']
\end{tikzcd}$$

Recall that ι is inclusion, so $\ker\phi = \ker q$. Because $\operatorname{im}\Gamma_* d_{c-1}$ and $\Gamma_* \operatorname{coker} d_{c-2}$ are equal in degree $d \geq -c+1$, $\ker\phi$ can only be nonzero in degree $-c$. This proves injection for degrees larger than c.

Lastly, we show injection in degree c. We have a short exact sequence

$$0 \to \ker\phi \to M \xrightarrow{\phi} (\Gamma_*\mathcal{F})_{\geq -c} \to 0.$$

Because $\ker\phi$ is only in degree $-c$, we can construct a module map $M \to \ker\phi$ to obtain a splitting by the composition

$$M \to M_{-c} \to \ker\phi$$

where $M \to M_{-c}$ is modding out all elements whose degrees are larger than $-c$ while $M_{-c} \to \ker\phi$ is just any projection of k-vector spaces. Therefore, we obtain

$$M \cong \ker\phi \oplus (\Gamma_*\mathcal{F})_{\geq -c}.$$

Exactness of (8) says that $L(P)$ is the minimal free resolution for M. Because $\ker\phi$ is just $\dim\ker\phi$ copies of k, uniqueness of minimal free resolution says that $L(P)$ must contain copies of the Koszul complex if $\ker\phi \neq 0$. However, $L(P)$ has length $c \leq n$, so the Koszul complex is not a summand of $L(P)$. This shows that ϕ is an isomorphism for degrees greater than or equal to $-c$, so (9) is indeed exact. $\square$

Remark 3.11. *Although Lemma 3.10 is stated using the BGG correspondence, one can write it without the BGG language as follows. Let*

$$\mathcal{C}_\bullet : \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-c)^{\oplus n_{-c}} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-c+1)^{\oplus n_{-c+1}} \longrightarrow \cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus n_0} \longrightarrow 0$$

be a resolution for some coherent sheaf $\mathcal{F}$ where $c \in \{0, 1, ..., n\}$. Then $\Gamma_*(\mathcal{C}_\bullet)$ is the minimal free resolution for the module $\Gamma_*(\mathcal{F})_{\geq 0}$

Lemma 3.10 allows us to check the indecomposability and simplicity of the BGG-sheaf from the indecomposability and simplicity of the module P.

Corollary 3.12. *Under the conditions of Lemma 3.10, a direct sum decomposition of $\mathcal{F}$ implies a direct sum decomposition of P.*

Proof. Suppose that $\mathcal{F}$ has a direct sum decomposition. Then $\Gamma_*(\mathcal{F})_{\geq c}$ has a direct sum decomposition as $M_1 \oplus M_2$. Because $L(P)$ is the minimal free resolution for $\Gamma_*(\mathcal{F})_{\geq c}$, $L(P)$ must be the sum of the minimal free resolutions for M_1 and M_2 due to the uniqueness of minimal free resolutions [9, Theorem 1.6]. Thus, one obtains $P = P_1 \oplus P_2$ by the equivalence of categories shown in Proposition 3.2. $\square$

Corollary 3.13. *Under the conditions of Lemma 3.10, if $\mathrm{Hom}_{EMod^{gr}}(P, P) = k$, then $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{F}) = k$.*

Proof. Suppose for contradiction that there exists $\phi \in \mathrm{Hom}(\mathcal{F}, \mathcal{F})$ such that $\phi \notin k$. Because the isomorphism $\mathcal{F} \to (\Gamma_*\mathcal{F})^\sim$ is natural, $\Gamma_*\phi \notin k$ where $\Gamma_*\phi : \Gamma_*(\mathcal{F})_{\geq -c} \to \Gamma_*(\mathcal{F})_{\geq -c}$ is the induced map of modules [15, Chapter 2 Proposition 5.15]. $\Gamma_*\phi$ then lifts to a map $\phi' \in \mathrm{Hom}(L(P), L(P))$ of free resolutions because free modules are projective. Note that $\phi' \notin k$ because $\Gamma_*\phi \notin k$. Due to the equivalence of category between linear free complexes of S-modules and graded E-modules (Proposition 3.2), ϕ' is a nontrivial morphism in $\mathrm{Hom}(P, P)$ so there is a contradiction. $\square$

4 Low Rank Examples

This section constructs two famous examples of low rank vector bundles: the Tango bundles and the null correlation bundle. Proposition 3.8 states that in order to produce vector bundles, one should construct faithful modules. By writing V in a basis that includes $v \in V$, it is easy to see that

$$\bigwedge^{l-1} V \xrightarrow{\cdot v} \bigwedge^{l} V \xrightarrow{\cdot v} \bigwedge^{l+1} V$$

is exact. Thus, a simple example of a faithful module is the truncation $E_{\leq c}$, i.e., the module obtained by deleting graded pieces whose degree is larger than c for $c = 0, 1, ..., n+1$. For instance,

$$\tilde{L}(E_{\leq 1}): \quad 0 \longrightarrow \mathcal{O} \longrightarrow V \otimes \mathcal{O}(1) \longrightarrow 0$$

is a resolution for the tangent bundle due to the Euler sequence.

Moreover, for a faithful module P and a general linear subspace $L \subset P_c$ where $c = \max_i(P_i \neq 0)$, P/L can be faithful again as long as the dimension of L is not too large.

Proposition 4.1. *Let P be a faithful module. Let*

$$b : P_{c-1} \times V \to P_c$$

be the bilinear map due to the E-module structure. Let k be such that $0 \leq k \leq \dim P_c - \dim \overline{\operatorname{im} b}$ where $\overline{\operatorname{im} b}$ denotes the Zariski closure of the image. Then for a general k-dimensional linear subspace $L \in \operatorname{Gr}(k, \dim P_c)$, P/L is faithful

Proof. For any $L \subset P_c$, $L \cap \operatorname{im} b = 0$ sufficiently shows that P/L is faithful: because P is already faithful, we just need to check the exactness of

$$P_{c-2} \xrightarrow{\cdot v} P_{c-1} \xrightarrow{\cdot v} P_c/L$$

at P_{c-1}, which is gauranteed if $L \cap \operatorname{im} b = 0$.

Note that $\operatorname{im} b \subset P_c$ being invariant under scaling implies $\overline{\operatorname{im} b}$ is too, so $\overline{\operatorname{im} b}$ can be viewed as the affine cone of some projective variety. Then a general linear subspace of dimension less than $\dim P_c - \dim \operatorname{im} b$ will be disjoint from $\operatorname{im} b$ [13, p.224]. □

Therefore, a valid strategy of constructing new bundles would be to exploit the faithfulness of $E_{\leq c}$ and apply Proposition 4.1 to form a quotient $E_{\leq c}/L$ that remains faithful. By quotienting the last non zero graded piece, one reduces the rank of the BGG-sheaf. In fact, the Tango bundles and the null correlation bundle can be constructed this way.

4.1 Tango Bundles

Consider the bilinear map:

$$b : V \times V \to \bigwedge^2 V.$$

The fiber $b^{-1}(b(u, v))$ where (u, v) is a general point in $V \times V$ is at least three dimensional because

$$b(u, v) = b(\lambda u, \frac{1}{\lambda} v) = b(u + \lambda v, v) = b(u, v + \lambda u)$$

for an arbitrary nonzero λ. In other words, the tangent space of the fiber are given by the three deformations shown above. Thus,

$$\dim \overline{\operatorname{im} b} \leq \dim(V \times V) - 3 = 2n - 1.$$

Due to Proposition 4.1, for a general $\binom{n+1}{2} - (2n-1)$-dimensional subspace $L \subset \bigwedge^2 V$, $E_{\leq 2}/L$ remains faithful. Proposition 3.8 implies that $\tilde{L}(E_{\leq 2}/L)$ provides a resolution for the BGG-sheaf $\mathcal{F}$, which in this case is a vector bundle of rank $n - 1$. Explicitly, the resolution is

$$0 \longrightarrow \mathcal{O} \longrightarrow V \otimes \mathcal{O}(1) \longrightarrow (\bigwedge^2 V/L) \otimes \mathcal{O}(2) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Proposition 4.2. *$\mathcal{F}$ is simple.*

Proof. The module producing $\mathcal{F}$ is $P = E_{\leq 2}/L$, which has a single generator 1. Thus, $\operatorname{Hom}(P, P) = k$. Corollary 3.13 tells us that $\mathcal{F}$ is simple. □

Evidently, twisting $\mathcal{F}$ will not change its simplicity nor its rank. The bundle $\mathcal{F}(-3)$ built from any such general L is called a Tango bundle [6].

Remark 4.3. *Traditionally, the existence of the Tango bundles required some argument on Chern classes while their simplicity was proved by manipulating exact sequences of sheaves to obtain $H^0(\mathbb{P}^n, \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{F}^\vee) = k$ [20, Section 4.3]. Here, Lemma 3.10 allows us to essentially just stare at the resolution and conclude that it must be simple. This approach is not in the literature as far as the author knows.*

4.2 Null Correlation Bundle

In this subsection, assume that n is odd. Let

$$\alpha := \sum_{i=0,2,\ldots,n-1} e_0 \wedge \ldots \wedge e_{i-1} \wedge \hat{e}_i \wedge \hat{e}_{i+1} \wedge e_{i+2} \wedge \ldots \wedge e_n$$

where $\hat{e}_i$ denotes ommision. Consider the $E-$module P defined by

$$P := \big(E/(\langle\alpha\rangle \oplus \wedge^n V \oplus \wedge^{n+1} V)\big)\,[n],$$

i.e., P is produced from quotienting out α as well as all elements in $\wedge^n V \oplus \wedge^{n+1} V$ from E and then shifting the grading so that the generator is in degree $-n$.

Proposition 4.4. *The BGG sheaf $\mathcal{F} = H^{-1}(\tilde{L}(P))$ is a vector bundle of rank $n-1$*

Proof. We will show $\mathcal{F}$ is a vector bundle by showing that P is a faithful module (the statement on rank is immediate). Because E is faithful, we know that

$$\cdots \to P_{i-1} \xrightarrow{\cdot v} P_i \xrightarrow{\cdot v} P_{i+1} \to \cdots$$

is exact for all v at $i \leq -3$. By definition $c = -1$ where $c = \max_i(P_i \neq 0)$. Thus, showing exactness at $i = -2$ for all $v \in V$, i.e.

$$\wedge^{n-3} V \xrightarrow{\cdot v} \wedge^{n-2} V \xrightarrow{\cdot v} (\wedge^{n-1} V)/\langle\alpha\rangle$$

being exact at $\wedge^{n-2} V$ sufficiently shows that P is faithful. Because $\wedge^{n-3} V \xrightarrow{\cdot v} \wedge^{n-2} V \xrightarrow{\cdot v} (\wedge^{n-1} V)$ is exact, knowing that $\alpha \notin b(\wedge^{n-2} V, V)$ where $b : \wedge^{n-2} V \times V \to \wedge^{n-1} V$ is the bilinear map defined by wedging sufficiently shows the exactness for $\wedge^{n-3} V \xrightarrow{\cdot v} \wedge^{n-2} V \xrightarrow{\cdot v} (\wedge^{n-1} V)/\langle\alpha\rangle$.

Linear algebra tells us that $\alpha \in \operatorname{im} b$ if and only if there exists a nonzero $v \in V$ such that $\alpha \wedge v = 0$. From the construction of α, we see that no such v can exist. $\square$

Proposition 4.5. *$\mathcal{F}$ is a simple vector bundle, i.e.* $\operatorname{Hom}(\mathcal{F}, \mathcal{F}) \cong \mathbb{C}$.

Proof. Immediate consequence of Corollary 3.13 $\square$

Applying the snake lemma to the exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker d_{-1} & \longrightarrow & \wedge^{n-1} V \otimes \mathcal{O}(-1) & \longrightarrow & \Omega^1(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker d_{-1} & \longrightarrow & (\wedge^{n-1} V/\alpha) \otimes \mathcal{O}(-1) & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}$$

gives us an exact sequence

$$0 \longrightarrow O(-1) \longrightarrow \Omega^1(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Dualizing this sequence produces the sequence that defines the null correlation bundle in [20, Section 4.2]. Thus $\mathcal{F}$ is the dual of the null correlation bundle.

5 Large Rank Bundles of Arbitrary Homological Dimensions

Compared to low rank bundles discussed before, bundles of rank greater than n on $\mathbb{P}^n$ are easier to construct. For instance, any bundle of rank r greater than n has quotient bundles of rank $n, n+1, ..., r$ [20, Lemma 4.3.1]. Thus, it is natural to demand more and ask whether large rank bundles with specific properties exist. The invariant that we are concerned with in this section is the homological dimension. This section will give a proof of Theorem 1.5 by constructing explicit bundles. Before the construction, we present some necessary facts about the homological dimensions of bundles.

5.1 Homological Dimension and Cohomology

A key property of the homological dimension is that it can be inferred from the cohomologies of a bundle and its twists (Proposition 1.4 of [3]).

Proposition 5.1. *For a vector bundle $\mathcal{F}$, $hd(\mathcal{F}) \leq d$ if and only if*

$$\bigoplus_{i \in \mathbb{Z}} H^q(\mathbb{P}^n, \mathcal{F}(i)) = 0$$

for all q such that $1 \leq q \leq n - d - 1$.

Remark 5.2. *By plugging in $d = n-1$, we see that $hd(\mathcal{F}) \leq n-1$ is always true. This is why the homological dimension for a vector bundle on $\mathbb{P}^n$ can only be $0, 1, ...,$ or $n - 1$.*

It is worth noting that throughout [3], the authors assumed k was characteristic 0. Nonetheless, the proof of Proposition 5.1 only involved performing a characteristic-blind induction with the base case being Horrock's splitting criterion. Given that Horrock's criterion holds in positive characteristics (Theorem 3.1 of [1]), Proposition 5.1 also holds.

Similar to the module case, we call a resolution of sheaves

$$\cdots \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_0 \longrightarrow 0$$

a linear resolution if each $\mathcal{F}_i$ is a direct sum of copies of $\mathcal{O}_{\mathbb{P}^n}(-i)$. Once a linear resolution is given for a vector bundle, no shorter resolution can exist.

Proposition 5.3. *If a vector bundle has a linear resolution of length l where $l \leq n-1$, then its homological dimension is l.*

Proof. Without loss of generality, $\mathcal{F}$ has a resolution

$$0 \longrightarrow \bigoplus \mathcal{O}_{\mathbb{P}^n}(-l) \longrightarrow \cdots \longrightarrow \bigoplus \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{F} \longrightarrow 0$$

By definition, the homological dimension is less than or equal to l. To show that it can be no less than l, we observe that $H^{n-l}(\mathbb{P}^n, \mathcal{F}(l-(n+1))) \neq 0$ by using the long exact sequence of cohomologies induced by the short exact sequence of sheaves inductively, so Proposition 5.1 gives us the result. $\square$

5.2 Construction of Simple Bundles

In this subsection we give the core construction needed to establish Theorem 1.5. From Corollary 3.13 and Proposition 5.3, we see that the problem of constructing simple bundles of homological dimension l relates to the problem of constructing simple faithful E-modules with l nonzero graded pieces. To construct simple modules, we introduce a convenient definition.

Definition 5.4. *Let U, W be vector spaces. Suppose that $L \subset U \otimes W$ satisfy the property that for all $\phi \in \operatorname{Hom}(U, U)$ such that $(\phi \otimes 1)(L) \subset L$, it is the case that $\phi \in k$ (i.e. ϕ is just a scaling). Then we say L* ***anchors*** *U.*

This notion is relevant for constructing simple modules due to the following.

Lemma 5.5. *Let P be the E-module defined by $P = P_0 \otimes_k \left(\bigoplus_{i=0}^{l} \bigwedge^i V\right)$ where P_0 is a k-vector space. Let $L \subset P_0 \otimes \bigwedge^l V$ be such that L anchors P_0. Then $\operatorname{Hom}_{EMod^{gr}}(P/L, P/L) = k$.*

Proof. Let $\phi : P/L \to P/L$ be a morphism of graded E-modules. Note that ϕ is completely determined by $\phi|_{P_0}$ because P/L is generated by P_0 as an E-module. Let $\tilde{\phi} : P \to P$ be the morphism determined by $\phi|_{P_0}$. Because ϕ preserves E-scalar multiplication, we have the following commutative square

$$\begin{array}{ccc} P & \xrightarrow{\tilde{\phi}} & P \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\ P/L & \xrightarrow{\phi} & P/L \end{array}$$

where q is the natural quotient map. The commutivity implies that

$$(\tilde{\phi}|_{P_0 \otimes \wedge^l V})(L) = (\phi|_{P_0} \otimes 1_{\wedge^l V})(L) \subset L.$$

Because L anchors P_0, $\phi|_{P_0} \in k$ so $\phi \in k$. $\square$

The above lemma tells us we can mod out by certain linear subspaces to construct simple E-modules. For our purpose of producing vector bundles, we need the resulting E-module to be faithful. Therefore, the following lemma is useful.

Lemma 5.6. *Let P be a faithful E-module and $l = \max_i P_i \neq 0$. Let*

$$\chi_i := \sum_{j=0}^{i} (-1)^{j-i} \dim P_j.$$

Let k be such that $0 \leq k \leq \chi_l - n$ (we assume that $\chi_l \geq n$ here). Then a general $L \in \mathrm{Gr}(k, P_l)$ has the property that P/L is a faithful module.

Proof. We have the bilinear map: $b : P_{l-1} \times V \to P_l$ defined by the E-action. For any $L \subset P_l$, $L \cap \operatorname{im} b = 0$ sufficiently shows that P/L is faithful: because P is already faithful, we just need to check the exactness of

$$P_{l-2} \xrightarrow{\cdot v} P_{l-1} \xrightarrow{\cdot v} P_l/L$$

at P_{l-1}, which is gauranteed if $L \cap \operatorname{im} b = 0$.

As a map of algebraic varieties, a general fiber F of b has $\dim F \geq \chi_{l-2} + 1$. This is because the map $\cdot v : P_{l-1} \to P_l$ has a kernel of dimension $\dim \ker(\cdot v) = \chi_{l-2}$ and there is the redundancy of relative scaling between P_{l-1} and V. Thus,

$$\begin{aligned} \dim \overline{\operatorname{im} b} \leq & \dim P_l + n + 1 - \chi_{l-2} - 1 \\ = & \chi_{l-1} + n \end{aligned}$$

where $\overline{\operatorname{im} b}$ denotes Zariski closure. Note that $\operatorname{im} b \subset P_l$ being invariant under scaling implies $\overline{\operatorname{im} b}$ is too, so $\overline{\operatorname{im} b}$ can be viewed as the affine cone of some projective variety. Then a general linear subspace of dimension less than $\dim P_l - \dim \operatorname{im} b \leq \chi_l - n$ will be disjoint from $\operatorname{im} b$ (p.224 of [13]). $\square$

The key linear algebra lemma that allows us to find anchoring linear subspaces is the following:

Lemma 5.7. *Let $n = \dim U > 1$, $m = \dim W \geq 4$ where U, W are finite dimensional k-vector spaces. Let d be an integer in the range $d \in (\frac{2n}{m}, mn - \frac{2n}{m})$. Then a general d-dimensional linear subspace $L \in \mathrm{Gr}(d, U \otimes W)$ anchors U.*

Note that this bound is not sharp but is sufficient for our purpose of constructing vector bundles. Before we prove Lemma 5.7, we present the construction for simple bundles of all homological dimensions.

Proof of Theorem 1.5. Fix a $l \in \{1, 2, ..., n-1\}$ and $r \geq n$. Let $p > 0$ be an integer so that

$$r < p\left(\binom{n}{l} - \frac{2}{\binom{n+1}{l}}\right).$$

Let $P_0 := k^p$ and

$$P := P_0 \otimes \left(\bigoplus_{i=0}^{l} \bigwedge^i V\right).$$

Because $\bigoplus_{i=0}^{l} \bigwedge^i V$ is faithful, P is also faithful. Simple calculation shows that $\chi_l = p\binom{n}{l}$. By applying Lemma 5.7 and Lemma 5.6, there exists a linear subspace $L \subset P_l = P_0 \otimes \bigwedge^l V$ with the following properties:

1. L anchors P_0,
2. P/L is a faithful module,
3. $\dim L = p\binom{n}{l} - r$

because $r \in \left[n, p\left(\binom{n}{l} - \frac{2}{\binom{n+1}{l}}\right)\right)$ implies that

$$p\binom{n}{l} - r \in \left(\frac{2p}{\binom{n+1}{l}}, p\binom{n+1}{l} - \frac{2p}{\binom{n+1}{l}}\right)$$

(using the identity $\binom{n+1}{l} - \binom{n}{l} = \binom{n}{l-1}$ allows us to see this). Let $\mathcal{F}$ be the BGG sheaf of P/L. Then we observe the following about $\mathcal{F}$:

- $\mathcal{F}$ is a vector bundle because P/L is faithful;
- $\mathcal{F}$ has rank r because $\dim L = p\binom{n}{l} - r$;
- $\mathcal{F}$ is simple because Lemma 5.5 tells us P/L is simple and because of Proposition 3.13;
- $\mathcal{F}$ has homological dimension l because of Proposition 5.3.

Therefore, we have constructed a vector bundle $\mathcal{F}$ of rank r and homological dimension l. □

Example 5.8. *We construct a simple bundle $\mathcal{F}$ of rank 5 and homological dimension 2 on $\mathbb{P}^3$ explicitly to illustrate the idea. Following the procedure described above, we consider the module*

$$P = k^2 \otimes \left(\bigoplus_{i=0}^{2} \bigwedge^i V\right)$$

and the quotient P/L where L is a general 1-dimensional subspace of $k^2 \otimes \left(\bigwedge^2 V\right)$. Lemma 5.6 and Lemma 5.7 then tells us that P/L is faithful and simple. Thus, $\tilde{L}(P/L)$ provides a resolution for its BGG-sheaf $\mathcal{F}$, *which is of rank* 5 *and homological dimension* 2*:*

$$0 \longrightarrow k^2 \otimes \mathcal{O} \longrightarrow (k^2 \otimes V) \otimes \mathcal{O}(1) \longrightarrow \left(\left(k^2 \otimes \bigwedge^2 V\right)/L\right) \otimes \mathcal{O}(2) \longrightarrow \mathcal{F} \longrightarrow 0.$$

The rest of this section is devoted to proving Lemma 5.7.

Proposition 5.9. *Let* U, W *be finite dimensional vector spaces with dimension* n, m *respectively. The subset of* d *dimensional subspaces that anchor* U *is Zariski open in* $\mathrm{Gr}(d, U \otimes W)$*.*

Proof. We can see this by setting up an incidence correspondence and applying upper semi-continuity. Consider

$$X \subset \mathbb{P}(\mathrm{End}(U \otimes W)) \times \mathrm{Gr}(d, U \otimes W)$$

such that $(\phi, L) \in X$ if and only if $\phi(L) \subset L$. Let $C \subset \mathrm{Gr}(d, U \otimes W)$ be the usual chart consisted of $d \times nm$ matrices whose first d by d minor is the identity. Then over C, X is given by the equation

$$\phi(L_i) \wedge L_1 \wedge L_2 \wedge ... \wedge L_d = 0$$

for $i = 1, ..., d$ where L_i is the vector represented by the ith row of the $d \times nm$ matrix corresponding to subspace L in chart C. Thus, X is closed in $\mathbb{P}(\mathrm{End}(U \otimes W)) \times C$. Because these charts cover all of the Grassmannian, X is a closed subset of $\mathbb{P}(\mathrm{End}(U \otimes W)) \times \mathrm{Gr}(d, U \otimes W)$. Consider $Z = \pi_1^{-1}(\mathbb{P}(\mathrm{End}(U))) \subset X$ where we are viewing $\mathbb{P}(\mathrm{End}(U)) \subset \mathbb{P}(\mathrm{End}(U \otimes W))$ as a linear subvariety using the identification $\phi \mapsto \phi \otimes 1$. Since $\pi_2 : Z \to \mathrm{Gr}(d, U \otimes W)$ is projective, we can apply the upper semi-continuity of fiber dimensions (Corollary 13.1.5 of [8]). L anchoring U is equivalent to the fiber Z_L being zero dimensional so upper semi-continuity gives us the proposition. □

Note that Proposition 5.9 does not say anything about the subset being nonempty, which is needed to give us Lemma 5.7.

Proposition 5.10. *Let* U, W *be finite dimensional vector spaces with dimension* n, m *respectively. A general* d *dimensional subspace of* $U \otimes W$ *anchors* U *if and only if a general* $nm - d$ *dimensional subspace anchors* U*.*

Proof. We can define

$$X^* \subset \mathbb{P}(\mathrm{End}(U^* \otimes W^*)) \times \mathrm{Gr}(nm - d, U^* \otimes W^*)$$

such that $(\psi, N) \in X^*$ if and only if $\psi(N) \subset N$. where U^* is the dual space of U. We define Z^* similarly using $\mathbb{P}(\mathrm{End}(U^*)) \subset \mathbb{P}(\mathrm{End}(U^* \otimes W^*))$. Let $f : Z \to Z^*$ be the map given by $(\phi, L) \mapsto (\phi^*, N)$ whre ϕ^* is the transpose of ϕ and N is the space of linear forms that vanish on L. It is clear that f is an isomorphism and the proposition follows. $\square$

We have a surjective rational map

$$(U \otimes W)^{\oplus d} \cong U \otimes W \otimes k^d \cong k^{nmd} \dashrightarrow \mathrm{Gr}(d, U \otimes W)$$

defined by $(v_1, ..., v_d) \mapsto v_1 \wedge ... \wedge v_d$. After fixing bases for U and W, we can write these data using indices conveniently. Let μ, ν be the indices for k^d, i, j for U and a, b for W. We can express v_μ by

$$v_\mu = v_{i\mu a} u^i \otimes w^a$$

where u^i, w^a are the basis vectors for U, W respectively and Einstein summation is used. Let $A \in \mathrm{End}(U)$. Let A^i_j be defined by $A(u^i) = A^i_j u^j$. Given $A \in \mathrm{End}(U)$, the condition that A fixes the subspace

$$A(\langle v_1, ..., v_d \rangle) \subset \langle v_1, ..., v_d \rangle$$

is equivalent to the existence of matrix C^ν_μ such that $A(v_\mu) = v_\nu C^\nu_\mu$. Using basis and indices, this is $A^i_j v_{i\mu a} u^j \otimes w^a = v_{i\nu a} C^\nu_\mu u^i \otimes w^a$, which can be written as

$$A^j_i v_{j\mu a} = v_{i\nu a} C^\nu_\mu \tag{10}$$

where it is understood that the equation applies for any $i = 1, ..., n$, $\mu = 1, ..., d$ and $a = 1, ..., m$. To summarize, we have the following:

Proposition 5.11. *Let U, W be finite dimensional vector spaces with dimension n, m respectively. If there exists an element $v_{j\mu a} \in U \otimes k^d \otimes W$ such that any pair of matrices (A^j_i, C^ν_μ) that satisfies (10) must be proportional to $(I_{n\times n}, I_{d\times d})$, then there exists an element $L \in \mathrm{Gr}(d, U \otimes W)$ that anchors U.*

Proof. Let $v_{j\mu a}$ satisfy the condition in the proposition. If the $v_1, ..., v_d$ constructed from $v_{j\mu a}$ are linearly independent, then L generated by them is an anchor because, as explained above, Equation 10 is just a rewriting of the condition for being an invariant subspace. However, if $v_1, ..., v_d$ are linearly dependent, we can find nonzero K^ν_μ such that $v_{i\nu a} K^\nu_\mu = 0$, so $(I_{n\times n}, (I_{d\times d} + K))$ is an alternative solution for Equation 10. Therefore, $v_1, ..., v_d$ must be linearly independent. Thus, $L = \langle v_1, ..., v_d \rangle$ will always anchor U. $\square$

To construct the tensor $v_{i\mu a}$ that satisfy Proposition 5.11, we will use the following fact from linear algebra.

Proposition 5.12. *There exist n by n matrices $B_1, B_2 \in M_{n\times n}$ such that any $C \in M_{n\times n}$ that commutes with both B_1, B_2 must be proportional to the identity.*

Proof. Let B_1 be a diagonal matrix with n different values on the diagonal. Then there are exactly 2^n subspaces of k^n that are invariant under B_1 given by the direct sums of the n distinct eigenspaces (p.811 of [5]). Consider n linearly independent vectors $x_1, ..., x_n$. Fix a subspace $U \subset k^n$ that is not 0 or k^n. The condition that no subset of $\{x_1, ..., x_n\}$ forms a basis for U is clearly a nonempty open condition for $\{x_1, ..., x_n\}$ because we can express it using the wedge product $\wedge$ and the Plücker embedding. Thus, the condition that no subset of $\{x_1, ..., x_n\}$ forms a basis for any of the $2^n - 2$ nontrivial invariant subspaces of B_1 is an nonempty open condition on the space of n linearly independent vectors. Pick $x_1, ..., x_n$ that satisfy this condition. Let B_2 be the matrix that has distinct eigenvalues for each of $x_1, ..., x_n$. Then, by construction, the only invariant subspaces shared by B_1, B_2 are 0 and k^n. Burnside's theorem on matrix algebra then says that B_1, B_2 generate $\mathrm{End}(k^n)$ [12]. Thus, any C that commutes with B_1, B_2 must commute with all matrices, so it has to be proportional to the identity. $\square$

Now we come to the key construction that will allow us to prove the subset of anchoring subspaces is nonempty.

Proposition 5.13. *Given $m \geq \max(\frac{n}{d}, \frac{d}{n}) + 2$, there exists $v_{i\mu a}$ such that the only solutions for Equation 10 are proportional to $(I_{n\times n}, I_{d\times d})$*

Proof. Equation 10 is a set of m different matrix equations:

$$\{AV_a = V_a C\}_{a=1,...,m}$$

where A, V_a, C are $n \times n$, $n \times d$, and $d \times d$ respectively. Because taking transpose is allowed, the roles of A and C are interchangeable as far this proposition is concerned. Thus, without loss of generality, we assume that $n \geq d$. In this case, if we prove the proposition for $m = \lceil \frac{n}{d} \rceil + 2$, any larger m will also work since they just add more constraints. Given a matrix M, let $M[r_1 : r_2 | c_1 : c_2]$ denote the $(r_2 - r_1) \times (c_2 - c_1)$ submatrix starting at row r_1 and column c_1.

For $a = 1, ..., \lceil \frac{n}{d} \rceil$, let V_a be defined by

$$V_a[(a-1)d + 1 : \min((a-1)d + 1 + d, n+1) | 1 : d+1] = I_{d\times d}[1 : \min(d, n - (a-1)d) + 1 | 1, d+1]$$

and zero everywhere else. Let V_{m-1}, V_m be be any $n \times d$ matrices such that

$$V_{m-1}[1 : d+1|1 : d+1] = B_1$$

$$V_m[1 : d+1|1 : d+1] = B_2$$

where B_is are the $d \times d$ matrices appearing in Proposition 5.12. Given that $AV_a = V_aC$ holds for $a = 1, ..., \lceil \frac{n}{d} \rceil$, due to V_a being the identity for specific rows and 0 elsewhere, we see that the matrix A must satisfy the following constraints:

- for $a = 1, ..., \lceil \frac{n}{d} \rceil$

$$A[(a-1)d+1 : \min((a-1)d+1+d, n+1)|(a-1)d+1 : \min((a-1)d+1+d, n+1)] \tag{11}$$

$$=C[1 : \min(d, n-(a-1)d)+1|1, 1 : \min(d, n-(a-1)d)+1] \tag{12}$$

- The other entries of A must be identically zero.

Given that $AV_a = V_aC$ holds for $a = m-1, m$, we see that $CB_i = B_iC$ for $i = 1, 2$. Thus, $C = \lambda I_{d\times d}$ for some $\lambda \in k$, so by Equation 11, we know $A = \lambda I_{n\times n}$. □

Now, we are finally ready for a proof of Lemma 5.7.

Proof of Lemma 5.7. Let us check that for $d \in (\frac{2n}{m}, \lfloor \frac{nm}{2} \rfloor]$, the condition of Proposition 5.13

$$m \geq \max(\lceil \frac{n}{d} \rceil, \lceil \frac{d}{n} \rceil) + 2$$

is met. For $d \in (\frac{2n}{m}, n]$, we have

$$\max(\lceil \frac{n}{d} \rceil, \lceil \frac{d}{n} \rceil) + 2 = \lceil \frac{n}{d} \rceil + 2 \leq \lceil \frac{nm}{2n} \rceil + 2 = \lceil \frac{m}{2} \rceil + 2 \leq m$$

when $m \geq 4$. For $d \in (n, \lfloor \frac{nm}{2} \rfloor]$, we have

$$\max(\lceil \frac{n}{d} \rceil, \lceil \frac{d}{n} \rceil) + 2 = \lceil \frac{d}{n} \rceil + 2 \leq \lceil \frac{nm}{2n} \rceil + 2 = \lceil \frac{m}{2} \rceil + 2 \leq m$$

when $m \geq 4$. Thus, by Proposition 5.11 and Proposition 5.9, for these values of d, a general d-dimensional subspace of $U \otimes W$ will anchor U. For $d \in (\lfloor \frac{nm}{2} \rfloor, nm - \frac{2n}{m})$, we simply apply Proposition 5.10, considering the case when nm is odd and the case when nm is even separately. □

6 BGG Correspondence and Cohomology Rings

Lastly, we turn to another interesting topic where the BGG correspondence shows up: the cohomology ring of Kähler manifolds. We will prove Proposition 1.6. Let X be any compact Kähler manifold. Then $H^1(X, \mathcal{O}_X)$ acts on $H^*(X, \mathcal{O}_X)$ by the cup product, making it a $\bigwedge H^1(X, \mathcal{O}_X)$-module. Futhermore, Theorem A of [19] states that given X does not carry any irregular fibrations (there is no map $f : X \to Y$ with positive dimensional fibers onto a normal analytic variety Y with the property that (any smooth model of) Y has maximal Albanese dimension), $H^*(X, \mathcal{O}_X)$ is faithful and the BGG-sheaf $\mathcal{F}$ it produces is a vector bundle of rank $\chi(\omega_X) := \sum_i (-1)^i h^i(X, \omega_X)$.

Assume for the rest of the section that $\dim X = 3$ and $q(X) := h^1(X, \mathcal{O}_X) \geq 5$. Then by Proposition 5.3, we see that $\mathrm{hd}(\mathcal{F}) = 3$. Thus, Proposition 1.2 of [17] then tells us that

$$\mathrm{rk}(\mathcal{F}) = \chi(\omega_X) \geq q(X) - 3.$$

Conjecture 3.9 of [19] in the case of threefolds states that $\chi(\omega_X) > q(X) - 3$ when $q(X)$ is large. Conjecture 3.9 proves to be very difficult. Nonetheless, something concrete can be said about the module $H^*(X, \mathcal{O}_X)$ suppose that the conjecture is not true: It must be generated by $1 \in H^0(X, \mathcal{O}_X)$. This is the content of Proposition 1.6. The rest of this section will be devoted to proving Proposition 1.6.

Because we will work purely on the module side and be agnostic to the fact that the module has a geometric origin, we rephrase Proposition 1.6 in as the following.

Proposition 6.1. *Suppose a $E = \bigwedge V$-module P satisfies all of the following:*

1. *The graded piece of P is given by*
$$P = \mathbb{C} \oplus V \oplus P_2 \oplus P_3;$$
2. *P is faithful;*
3. *$-\chi(P) < n - 1$.*

Then P is generated in degree 0.

To prove this, we will first need some propositions.

Proposition 6.2. *Let $b : U \times V \to W$ be any bilinear map such that $b(-, v)$ has the same rank for all $v \in V$. Then $[\mathrm{im}\, b] \subset \mathbb{P}W$ is a projective subvariety.*

Proof. Due to having constant rank, we have an induced map $f : \mathbb{P}V \to \mathrm{Gr}(r, W)$ where r is the rank of $b(-, v)$ for any $v \in V$. $\mathrm{im}\, f$ is a projective variety since f is a closed map. We can form the incidence correspondence $X \subset \operatorname{im} f \times \mathbb{P}W$ defined by $(H, w) \in X$ if and only if $w \in H$. Then $[\operatorname{im} b] = \pi_2(X)$ must be a closed subvariety of $\mathbb{P}W$. $\square$

Proposition 6.3. *Let $K = K_0 \oplus K_1$ be a faithful E module with two graded pieces. Let $b : K_0 \times V \to K_1$ be the multiplication map. Then for a general $p = (k, v) \in K_0 \times V$, the fiber $F_p := b^{-1}(b(p))$ is one dimensional and smooth at p. Moreover, $T_pF_p = \langle (k, -v) \rangle \subset T_p(K_0 \times V) = K_0 \oplus V$.*

Proof. If $\dim K_1 < \dim K_0 + n$, then the BGG complex $\tilde{L}(K)$ produces a vector bundle of rank $r < n$ but with homological dimension 1, which is impossible due to Proposition 1.2 of [17]. Thus, $\dim K_1 \geq \dim K_0 + n$.

Suppose for contradiction that $\dim imb < \dim K_0 + n$. Then Proposition 6.2 tells us we can find linear subspace $H \subset K_1$ satisfying $\dim H = \dim K_1 - \dim imb > \dim K_1 - \dim K_0 - n$ and $H \cap \operatorname{im} b = 0$. Then $K/H = K_0 \oplus K_1/H$ remains faithful and produces a bundle of rank less than n and homological dimension 1, which is impossible due to Proposition 1.2 of [17]. Thus, $\dim \operatorname{im} b \geq \dim K_0 + n$. Note that $\dim \operatorname{im} b \geq \dim K_0 + n + 1$ is impossible because a bilinear map has general fibers that are at least 1-dimensional due to the freedom of relative scaling. Thus, $\dim \operatorname{im} b = \dim K_0 + n$ and the general fiber must be 1-dimensional.

The rest of the proposition follows from generic smoothness. The explicit generator for the tangent space of the fiber is exactly the tangent vector that generates relative scaling between K_0 and V. $\square$

Proof of Proposition 6.1. Suppose for contradiction that P has a degree 3 generator. Then, stripping off that degree 3 generator produces a faithful module whose BGG-sheaf is a vector bundle of rank less than $n - 2$ and whose homological dimension is 3. This is impossible by Corollary 1.7 of [3].

Therefore, all there is to prove is that $\phi : \wedge^2 V \to P_2$ induced by the E-module structure is surjective. Suppose for contradiction that ϕ is not surjective. Let $K_2 \subset P_2$ be any linear subspace such that $K_2 \to P_2 \to P_2/\operatorname{im} \phi$ is an isomorphism. Then K_2 is non trivial. Let

$$K := \langle K_2 \rangle = K_2 \oplus K_3$$

$$Q := \langle \mathbb{C} \rangle = \mathbb{C} \oplus V \oplus Q_2 \oplus Q_3$$

be the submodule generated by the degree 2 and degree 0 generators respectively. We observe that $P_2 = K_2 \oplus Q_2$. Thus, $P = (K \oplus Q)/L$ where $L \subset K_3 \oplus Q_3$ is some linear subspace. Moreover, the faithfulness of P implies the faithfulness of K and Q. Because P is faithful, it must be the case that $L \cap \operatorname{im} b = 0$ where

$b : (K_2 \oplus Q_2) \times V \to K_3 \oplus Q_3$ is the natural bilinear map due to the module structure. We have

$$\begin{aligned} -\chi(P) &= -\chi(K) - \chi(Q) - \dim L \\ &= \dim(K_3 \oplus Q_3) - \dim L - \dim(K_2 + Q_2) + n < n - 1. \end{aligned}$$

We know that $\dim L + \dim \operatorname{im} b \leq \dim(K_3 \oplus Q_3)$ because Proposition 6.2 tells us that $\operatorname{im} b$ is the affine cone of a projective variety and $L \cap \operatorname{im} b = 0$. Thus, we have

$$\dim \operatorname{im} b \leq \dim(K_3 + Q_3) - \dim L < \dim(K_2 + Q_2) - 1.$$

Let $p = (k, q, v) \in (K_2 \oplus Q_2) \times V$ be a general point. $F_p := b^{-1}b(p)$ is smooth at p and is of dimension $\dim F_p > n + 2$ by applying generic smoothness. By the definition of b, it is clear that $(0, v \wedge V, 0) \oplus \langle (k, q, -v) \rangle \subset T_pF_p \subset (K_2 \oplus Q_2) \times V$. This known subspace already takes up exactly $n+1$ dimensions of the tangent space of the fiber. Because $\dim F_p > n+1$, we have an extra $(k', q', v') \in T_pF_p$ linearly independent from $(0, v \wedge V, 0) \oplus \langle (k, q, -v) \rangle$. Notice that we have projection $\pi : (K_2 \oplus Q_2) \times V \to K_2 \times V$ so that $\pi(F_p) \subset F_p^K$ where F_p^K is the fiber for $b^K : K_2 \times V \to K_3$ at (k, v). Then, $d\pi_p(k', q', v') = (k', v') \in T_pF_p^K = \langle (k, -v) \rangle$ by Proposition 6.3. In other words, $(k', v') = \lambda(k, -v)$ for some nonzero λ. Subtracting off $\lambda(k, q, -v)$ from (k', q', v') then produces some $(0, q'', 0) \in T_pF_p$ where $q'' \in Q_2$ and $q'' \notin v \wedge V$.

We have $db_p(0, q'', 0) = 0$ due to $(0, q'', 0)$ being in the fiber direction. Because b is a bilinear map, $b(q'', v) = db_p(0, q'', 0) = 0$. This violates the faithfulness of Q and we have a contradiction.

$\square$

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